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Effective potential for uniform magnetic fields through Pauli interaction

Hyun Kyu Lee and Yongsung Yoon

Department of Physics, Hanyang University,
Seoul 133-791, Korea
E-mail: `hyunkyulee@hanyang.ac.kr`, `cem@hanyang.ac.kr`

Abstract: We have calculated the explicit form of the real and imaginary parts of the effective potential for uniform magnetic fields which interact with spin-1/2 fermions through the Pauli interaction. It is found that the non-vanishing imaginary part develops for a magnetic field stronger than a critical field, whose strength is the ratio of the fermion mass to its magnetic moment. This implies the instability of the uniform magnetic field beyond the critical field strength to produce fermion pairs with the production rate density $w(x) = \frac{m^4}{24\pi}(\frac{|\mu B|}{m} - 1)^3(\frac{|\mu B|}{m} + 3)\theta(\frac{|\mu B|}{m} - 1)$ in the presence of Pauli interaction.

Keywords: Nonperturbative Effects, Integrable Field Theories.

Contents

1. Introduction

The interaction of charged spin-1/2 fermions with electromagnetic fields is described by the minimal coupling in the form of the Dirac equation. One of the interesting phenomena with strong electromagnetic fields is the particle production. A well known example is the Schwinger process, in which minimally interacting charged particles are created in pairs in strong electric fields [1, 2]. In a pure magnetic field configuration, however, it has been shown that the production of minimally interacting fermion is not possible even with a spatial inhomogeneity [3]. Therefore, the pair production of minimally interacting particles is considered to be a purely electric effect.

Pauli introduced a non-minimal coupling of spin-1/2 particles with electromagnetic fields, which can be interpreted as an effective interaction of fermions with an anomalous magnetic moment [4 − 6]. For the neutral fermions with non-vanishing magnetic moments, it is the Pauli interaction through which the electromagnetic interaction can be probed. It is interesting to note that the inhomogeneity of the magnetic field, which couples directly to the magnetic dipole moment through the Pauli interaction, plays a similar role analogous to the electric field for a charged particles with the minimal coupling. The possibility of production of the neutral fermions in a pure magnetic field configuration with spatial inhomogeneity has been demonstrated in 2+1 dimension [7], and recently the production rate in 3+1 dimension has been calculated explicitly for the magnetic fields with a spatial inhomogeneity of a critical value [8].

The purpose of this paper is to discuss further the possibility of the fermion production under a uniform magnetic field when it becomes stronger than the critical field whose strength is the ratio of the fermion mass to its magnetic moment. We consider a neutral fermion but with a magnetic moment μ with Pauli interaction. The energy eigenvalues of the fermion [9] are given by

$$E = \pm\sqrt{p_l^2 + \left(\sqrt{m^2 + p_t^2} - |\mu B|\hat{s}\right)^2}, \tag{1.1}$$

where p_l and p_t are respectively the longitudinal and the transversal momentum to the magnetic field direction, and $\hat{s} = \pm 1$ are spin projections along the magnetic field. One can see that, for a critical magnetic field $B_c = \frac{m}{\mu}$, the energy gap between the positive and the negative energy states disappears. This indicates the possible instability of magnetic field configurations even in uniform magnetic fields. The generic feature of the instability due to the level crossing of the lowest energy state is the appearance of an imaginary part in the effective potential for the background field [10, 11]. We have calculated the effective potential of uniform magnetic fields which interact with spin-1/2 fermions through the Pauli interaction. For a magnetic field weaker than the critical field, we obtain the real effective potential as expected. However, for a magnetic field stronger than the critical field, it is found that the imaginary part of the effective potential does not vanish. This implies that a uniform magnetic field becomes unstable to produce the fermion pairs in vacuum when it is stronger than the critical field. It should be noted that the pair production in uniform magnetic fields is not due to the tunnelling process as in Schwinger process overcoming the energy gap, $2m$, but due to the disappearance of the energy gap in eq. (1.1) for the critical field strength. The difference is also manifested in different functional forms of the pair production rates. It is found that the production rate takes a quartic form which is quite different from the exponential form of the Schwinger process.

The calculation of the effective potential for uniform magnetic fields induced by a neutral fermion, which is assumed to be interacting with the background electromagnetic field through the Pauli coupling, is discussed in section II and the results are summarized in section III.

2. Effective potential for uniform magnetic fields induced by fermions with Pauli interaction

The Dirac Lagrangian of a neutral fermion with the Pauli interaction is given by

$$\mathcal{L} = \bar{\psi}\left(\not{p} + \frac{\mu}{2}\sigma^{\mu\nu}F_{\mu\nu} - m\right)\psi, \tag{2.1}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, $g_{\mu\nu} = (+,-,-,-)$. μ in the Pauli term measures the magnitude of the magnetic moment of the neutral fermion. The corresponding Hamiltonian is given by

$$H = \vec{\alpha}\cdot(\vec{p} - i\mu\beta\vec{E}) + \beta(m - \mu\vec{\sigma}\cdot\vec{B}), \tag{2.2}$$

where $\sigma^i = \frac{1}{2}\epsilon^{ijk}\sigma^{jk}$. The energy eigenvalues eq. (1.1) of the Hamiltonian eq. (2.2) are obtained diagonalizing the 4×4 Pauli Hamiltonian for a constant $\vec{B}$ with $\vec{E} = 0$. One can see that, for a magnetic field stronger than the critical field $B_c = \frac{m}{\mu}$, the energy gap between the positive and the negative energy states disappears. This indicates the possible instability of magnetic field configuration.

On the other hand, the energy eigenvalues of minimally interacting charged fermions without an anomalous magnetic moment are

$$E = \pm\sqrt{p_l^2 + m^2 + |eB|(2n + 1 - \mathrm{sgn}(e)\hat{s})}, \tag{2.3}$$

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where $n = 0, 1, 2, \ldots$ [12]. It should be pointed out that zero energy states do not exist even for a strong magnetic field, and no particle production of minimally interacting fermions in pure magnetic fields can be attributed to this finite energy gap. It is also interesting to note that the energy eigenvalues for a uniform color-magnetic field $\mathcal{B}$ configuration in a pure Yang-Mills theory are given by

$$E = \pm\sqrt{p_l^2 + |g\mathcal{B}|(2n + 1 - 2\hat{s})}, \tag{2.4}$$

where the factor 2 in front of $\hat{s}$ is due to the spin $S = 1$ of the gauge fields. One can see that eq. (2.4) shows a level crossing for the state with $n = 0$ and $\hat{s} = +1$. In fact, the instability of the uniform color-magnetic field configuration due to the level crossing has been discussed by Nielsen and Olesen [11] in detail, where a pure Yang-Mills theory is shown to be unstable for a massless non-uniform field excitation known as Nielsen-Olesen mode and they found the non-vanishing imaginary part of the corresponding effective potential in a quadratic form of the fields.

The effective potential $V_{\rm eff}(A)$ for a background electromagnetic vector potential A_μ can be obtained by integrating out the fermions:

$$-i\int d^4x V_{\rm eff}(A[x]) = \int d^4x \left\langle x \left| tr \ln\left[\left(\not{p} + \frac{\mu}{2}\sigma^{\mu\nu}F_{\mu\nu} - m\right)\frac{1}{\not{p} - m}\right]\right| x\right\rangle, \tag{2.5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and tr denotes the trace over Dirac algebra. The decay probability of the background magnetic field into the neutral fermions is related to the imaginary part of the effective potential $V_{\rm eff}(A)$,

$$P = 1 - |e^{i\int d^4x V_{\rm eff}(A[x])}|^2 = 1 - e^{-2\Im\int d^3x dt V_{\rm eff}(A[x])}. \tag{2.6}$$

That is, the twice of the imaginary part of the effective potential $V_{\rm eff}(A[x])$ is the fermion production rate per unit volume [13]: $w(x) = 2\Im(V_{\rm eff}(A[x]))$ for small probabilities.

For a uniform magnetic field configuration, $\vec{B} = B\hat{z}$, the integral form of the effective potential eq. (2.5) is obtained as [8]

$$V_{\rm eff} = -\frac{(\mu B)^2}{4\pi^2}\int_0^\infty \frac{ds}{s^2}\left[i\int_0^1 d\xi(1-\xi)e^{i(\mu B)^2\xi^2 s} - \frac{i}{2} + \frac{(\mu B)^2 s}{12}\right]e^{-im^2 s}. \tag{2.7}$$

The integration eq. (2.7) can be done explicitly. Introducing dimensionless parameters, $t = m^2 s$ and $\beta = \frac{|\mu B|}{m}$, the imaginary part of the effective potential eq. (2.7) can be written as

$$\begin{aligned}\Im(V_{\rm eff}) &= -\frac{m^4\beta^2}{4\pi^2}\int_0^1 d\xi(1-\xi)\int_0^\infty \frac{dt}{t^2}\left[\cos(\beta^2\xi^2 - 1)t - \cos(t) - \beta^2\xi^2 t\sin(t)\right] \\ &= -\frac{m^4\beta^2}{8\pi}\int_0^1 d\xi(1-\xi)[1 - \beta^2\xi^2 - |1 - \beta^2\xi^2|]. \end{aligned} \tag{2.8}$$

For a magnetic field weaker than the critical field, $\beta \leq 1$, the integration eq. (2.8) vanishes. It can be also verified by a contour integration. For the magnetic fields weaker than the critical field $B_c = m/\mu$, using a contour integration in the fourth quadrant, the

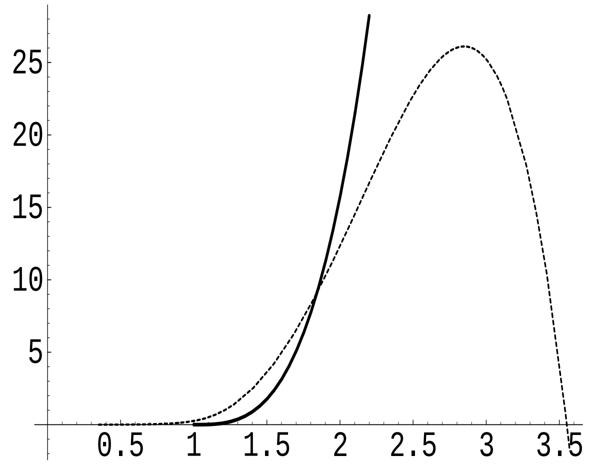

Figure 1: Effective potential of the uniform magnetic field B induced by neutral fermions with a magnetic moment: vertical axis is $V_{\rm eff}$ in the unit of $\frac{m^4}{48\pi^2}$ (the solid line is for the imaginary part and the dashed line is for the real part), horizontal axis is $\beta(=|\mu B|/m)$.

integration can be done along the negative imaginary axis giving the finite real effective action as

$$V_{\rm eff} = -\frac{(\mu B)^2}{4\pi^2}\int_0^\infty \frac{ds}{s^2}\left[\frac{1}{2}+\frac{(\mu B)^2 s}{12}-\int_0^1 d\xi(1-\xi)e^{(\mu B)^2\xi^2 s}\right]e^{-m^2 s}. \tag{2.9}$$

Therefore, one can see that the uniform magnetic fields weaker than the critical field are stable as expected.

However, for a magnetic field stronger than the critical field, $\beta > 1$, the imaginary part of the effective potential does not vanish, but takes a quartic form:

$$\Im(V_{\rm eff}) = \frac{1}{48\pi}(|\mu B|-m)^3(|\mu B|+3m)\theta(|\mu B|-m). \tag{2.10}$$

This result shows that the uniform magnetic fields stronger than the critical field, $B_c = m/\mu$, are unstable and reduce the field strengths by producing the fermion pairs with the rate, $w(x) = 2\Im(V_{\rm eff})$.

The real part of the effective potential eq. (2.7) can be calculated explicitly as well,

$$\begin{aligned}\Re(V_{\rm eff}) &= -\frac{m^4\beta^2}{4\pi^2}\int_0^1 d\xi(1-\xi)\int_0^\infty \frac{dt}{t^2}\left[\sin(1-\beta^2\xi^2)t-\sin(t)+\beta^2\xi^2 t\cos(t)\right] \qquad (2.11)\\ &= \begin{cases}\frac{m^4}{288\pi^2}[13\beta^4-78\beta^2+96\beta\tanh^{-1}(\beta)-6(\beta^4-6\beta^2-3)\ln(1-\beta^2)], \text{ for } \beta\le 1\\ \frac{m^4}{288\pi^2}[13\beta^4-78\beta^2+96\beta\coth^{-1}(\beta)-6(\beta^4-6\beta^2-3)\ln(\beta^2-1)], \text{ for } \beta\ge 1.\end{cases}\end{aligned}$$

For a weak field, $\beta \ll 1$, eq. (2.11) approximates to $\frac{(\mu B)^6}{240\pi^2 m^2}$, and for the critical field, $\beta = 1$, $\Re(V_{\rm eff}) = (96\ln 2 - 65)\frac{m^4}{288\pi^2}$. The real and imaginary parts of the effective potential with respect to the magnetic field strength are shown in FIG.1 in the unit of $\frac{m^4}{48\pi^2}$.

So far, we have considered only the neutral fermions with a magnetic dipole moment. It is also interesting to see how the instability due to the Pauli interaction is affected when the minimal coupling is turned on in addition to the Pauli interaction. Let us consider an effective Lagrangian, which might describe a fermion endowed with a non-vanishing electric charge e and as well as with a magnetic dipole moment μ, given by

$$\mathcal{L} = \bar{\psi}(\not{p} - e\not{A} + \frac{\mu}{2}\sigma^{\mu\nu}F_{\mu\nu} - m)\psi. \tag{2.12}$$

In this work, we consider a model in which the electric charge e and the magnetic dipole moment μ are two independent couplings such that the Pauli term description of the magnetic moment is valid up to the critical magnetic field.[1] Then the effective potential for a fermion described by eq. (2.12) is calculated as [17]

$$\begin{aligned} V_{\text{eff}} = &- \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^2}\left[|eB|\coth(|eB|s) - \frac{1}{s} - \frac{(eB)^2 s}{3}\right]e^{-m^2 s} \\ &- \frac{(\mu B)^2}{4\pi^2}\int_0^\infty \frac{ds}{s^2}\left[i|eB|s\cot(|eB|s)\int_0^1 d\xi(1-\xi)e^{i(\mu B)^2\xi^2 s} - \frac{i}{2} + \frac{(\mu B)^2 s}{12}\right]e^{-im^2 s}, \end{aligned} \tag{2.13}$$

where the first integral is the well-known effective potential for a minimally interacting charged fermion in a uniform magnetic field [3], and the second integral is the contribution from the magnetic moment μ.

For a magnetic field weaker than the critical field $B_c = m/\mu$, the s integration of the second integral can be performed along the negative imaginary axis in the fourth quadrant. Thus, the effective potential eq. (2.13) can be written as

$$\begin{aligned} V_{\text{eff}} = &- \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^2}\left[|eB|\coth(|eB|s) - \frac{1}{s} - \frac{(eB)^2 s}{3}\right]e^{-m^2 s} \\ &- \frac{(\mu B)^2}{4\pi^2}\int_0^\infty \frac{ds}{s^2}\left[\frac{1}{2} + \frac{(\mu B)^2 s}{12} - |eB|\coth(|eB|s)\int_0^1 d\xi(1-\xi)e^{(\mu B)^2\xi^2 s}\right]e^{-m^2 s}, \end{aligned} \tag{2.14}$$

which is real as expected. For a small μ, expanding this exact effective potential eq. (2.14) in terms of μ, the leading contribution is given by

$$\begin{aligned} V_{\text{eff}} \cong &- \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^2}\left[|eB|\coth(|eB|s) - \frac{1}{s} - \frac{(eB)^2 s}{3}\right]e^{-m^2 s} \\ &+ \frac{(\mu B)^2}{8\pi^2}\int_0^\infty \frac{ds}{s^2}[|eB|\coth(|eB|s) - 1]e^{-m^2 s}, \quad \text{for } \left|\frac{\mu B}{m}\right| \ll 1, \end{aligned} \tag{2.15}$$

which is equivalent to the perturbative result calculated up to the leading order of μ with $\mu = \mu_a$ in [15]. However, it should be noted that eq. (2.15) can not be applicable for strong magnetic fields, for which $|\frac{\mu B}{m}| \ll 1$ is not valid.

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[1] In [14, 15], μ was identified as the Schwinger's anomalous magnetic moment $\mu_a = \frac{\alpha}{2\pi}\frac{e}{2m}$. However, the calculations of the 1-loop QED radiative corrections for strong magnetic fields [16] show that the Pauli interaction of anomalous magnetic moment μ_a is only valid for weak field limit, but not for magnetic fields stronger than m^2/e which is much weaker than the critical field, $B_c = m/\mu_a$.

For a magnetic field stronger than the critical field, isolating singularities at $s = 0$ in the second integral of eq. (2.13), we can rewrite the effective potential as

$$
\begin{aligned}
V_{\text{eff}} = &- \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^2}\left[|eB|\coth(|eB|s) - \frac{1}{s} - \frac{(eB)^2 s}{3}\right]e^{-m^2 s} \\
&- \frac{(\mu B)^2}{4\pi^2}\int_0^\infty \frac{ds}{s^2}\left[i\int_0^1 d\xi(1-\xi)e^{i(\mu B)^2\xi^2 s} - \frac{i}{2} + \frac{(\mu B)^2 s}{12}\right]e^{-im^2 s} \\
&+ \frac{(\mu B)^2}{4\pi^2}\int_0^\infty \frac{ds}{s^2}\{|eB|s\coth(|eB|s) - 1\} \\
&\quad \left[\int_0^{\xi_0} d\xi(1-\xi)e^{-(m^2-(\mu B)^2\xi^2)s} - \int_{\xi_0}^1 d\xi(1-\xi)e^{-((\mu B)^2\xi^2-m^2)s}\right], \qquad (2.16)
\end{aligned}
$$

and $\xi_0 \equiv |\frac{m}{\mu B}| < 1$. In eq. (2.16), the first integral is known to be real, and it is straightforward to verify that the third integral is real. The second integral in eq. (2.16) is exactly the effective potential eq. (2.7) obtained for neutral fermions with the Pauli interaction. It shows that for a fermion described by eq. (2.12) the imaginary part of the effective potential comes only from the contributions of the magnetic moment through Pauli interaction and the instability for $B \geq B_c$ is not affected by the electric charge with the minimal coupling.

Therefore, regardless of whether the fermion is charged or neutral, the fermion carrying a magnetic moment which interacts with Pauli term causes the instability of uniform magnetic field configuration when the field strength is larger than the critical value, which is determined by the ratio of the fermion mass to the anomalous magnetic moment.

3. Discussion

For charged fermions, which couple to the electromagnetic field through the minimal coupling, it has been well known that the pair creation is not possible in pure magnetic field configurations [1, 3, 12]. In this work, we discuss the possibility that particles can be created in a strong enough magnetic field as a purely magnetic effect. Introducing the magnetic moment of neutral spin-1/2 fermions through the Pauli interaction, it has been shown that the production of neutral fermions is possible in a pure magnetic field configuration provided that the gradient of the magnetic field is extremely strong [7, 8]. However, the particle production in uniform magnetic fields has not yet been addressed properly. We calculate explicitly the real and imaginary part of the effective potential for a uniform magnetic field, by integrating out the fermions with a magnetic moment which couples to the magnetic fields through the Pauli interaction. We have shown explicitly that the imaginary part of the effective potential develops when the uniform magnetic fields are stronger than the critical field $B_c = \frac{m}{\mu}$. Hence the magnetic field background stronger than B_c is unstable to produce the fermion pairs. We have calculated the production rate density w of the fermions as $w = \frac{m^4}{24\pi}(\frac{|\mu B|}{m} - 1)^3(\frac{|\mu B|}{m} + 3)\theta(\frac{|\mu B|}{m} - 1)$. One can note that this result is quite different from the exponential form of the Schwinger process. The main reason for this difference is that the pair production in uniform magnetic fields is not due to the tunnelling process as in the Schwinger process overcoming the energy gap $2m$, but due to the disappearance of the energy gap.

One of the immediate application of the particle production mechanism discussed in this work might be a particle creation in the vicinity of the compact objects in the strong explosive astrophysical phenomena, where the extraordinarily strong magnetic fields ($> 10^{15}$G) are expected and the environment is considered to be magnetically dominant. Of course it depends on whether there is any physical model in which the neutral fermion considered is described by Pauli interaction up to the critical magnetic fields.

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